SYNTHESIS OF AN OPTIMUM IMPEDANCE TRANSFORMER

Henry J. Riblet
Microwave Development Laboratories, Inc.
Needham, Massachusetts

Abstract

This paper considers the problem of optimizing the match between a generator and a resistive load by means of a transformer consisting of equal length line sections alternating with the same length, shunted, shorted stubs. It is argued that, in the optimum transformer, only that stub appears which is in shunt with the low impedance termination. General design formulas are given for a two section transformer with equi-ripple performance over the design band. For an octave band and a two to one transformation, it is shown that this stepped, shunted transformer has substantially superior performance to that of a conventional stepped transformer of the same length. Detailed computations are made in this case which indicate the extent to which this equi-ripple design is optimum.

Introduction

The interest in the optimum design of a distributed transformer between two impedances goes back for many years. Optimum performance has been shown for stepped transformers with Tchebyscheff performance on one hand and for various continuous transformers on the other, depending on the particular realization allowed. In general, the discussion of optimum distributed transformers has been limited to monotonic transformers. This paper is concerned, however, with the synthesis of a class of optimum transformers which are not monotonic in any general sense; but are physically realizable, nevertheless, for 2:1 impedance transformations covering an octave band in one half a wavelength. They are absolutely optimum in the sense that no other form of transformer is known which will perform as well in so short a length.

Consider a matching transformer between a load impedance, R_o, assumed to be less than one and a generator of unity impedance which consists of a cascade of line sections each a quarter of a wavelength long at the mid-band frequency, with different characteristic impedances, together with a series of short circuited stubs also a quarter of a wavelength long, having different characteristic admittances, and each placed in parallel with the equal length line sections at the points where they join. It will be convenient to call such a transformer a shunted, stepped transformer to distinguish it from the familiar stepped transformer which lacks the shunt stubs. The stub which is directly in parallel with the load, R_o, plays an especially important role in the discussion and its admittance will be denoted by Y_o.

It was the original objective of this investigation to design an optimum 2:1 transformer to operate over approximately an octave frequency band. It was soon discovered that, with the type of transformer under consideration, a VSWR $<$ 1.03 over the octave band could be achieved with a two-step prototype transformer. For this reason, the analysis to be presented will, in the interest of clarity, be limited to this simple case even though most of the conclusions apply in general.

A Simplifying Theorem

A further analytical simplification is provided by a theorem which states, that, having fixed the band width of the transformer and the value of R_o, the only stub that appears in the optimum transformer immediately precedes R_o. This result is the immediate consequence of the fact that the load resistance in such a cascade always decreases as Karoda's identities are used to transfer the shunt stubs to the end of cascade and the intuitively obvious, but unproved theorem that the performance of the optimum, shunted, stepped transformer for a fixed R_o must always deteriorate as R_o decreases. Thus if the optimum transformer for a certain R_o contained stubs not at the end of the cascade, Karoda's identities could be used to move them to the end. The result would be a shunted, stepped transformer which matches a load of resistance less than R_o as well as R_o can be matched. This is intuitively impossible. Thus in determining the optimum, two-section transformer, only Y_o and the characteristic impedances, Z_1 and Z_2, of the sections have to be determined.

The Design Procedure

If the frequency variable, t, is used, where $t = \cot\theta$ and $\theta = 2\pi L/L_o$, then it is easily shown that the inertion loss function, P_L, of the network is given by

$$P_L = 1 + (A^2 + B^2)/4R_o(1 + t^2)^2 \qquad (1)$$

where

$$A = [R_oY_o(Z_1 + Z_2) + R_o - 1]t^2 + Z_1/Z_2 - R_oZ_2/Z_1$$

and

$$B = R_oY_ot^3 - [R_o(Y_oZ_1/Z_2 + 1/Z_1 + 1/Z_2) - Z_1 - Z_2]t.$$

The problem of optimizing the performance of the transformer is thus the problem of minimizing the value of the fraction in (1) over the frequency band of interest.

If we put $Y_o = 0$ and require that $R_o = Z_1Z_2$, then $B \equiv 0$; and it is found that the constant term in A can be chosen so that $A/(1 + t^2)$ is equi-ripple over the design band by the proper choice of Z_1 and Z_2. This is the well known optimum design procedure for an equal length stepped transformer. Stated in a slightly different way, the given function, $(R_o - 1)t^2/2\sqrt{R_o}(1 + t^2)$ has been approximated in such a way that the well-known Tchebyscheff criterion for an optimum approximation are satisfied. It is a point of interest that, for $R_o = .5$, such a transformer will have a VSWR $\approx$ 1.13 over an octave band.

On the other hand, if $Y_o \neq 0$, there are three unknowns with which to make $A \equiv 0$ and arrange so that $B/(1 + t^2)$ is equi-ripple over the design band. In fact, the three simultaneous equations can be solved explicitly for Y_o, Z_1 and Z_2; and then the maximum VSWR over the design band is readily found. For the particular example under consideration, the approximate values of Y_o, Z_1, Z_2 are .66, .62 and .88 respectively, while the maximum VSWR $<$ 1.026 over the octave band. The fact that the introduction of a stub in front of the load actually improves the performance of the stepped transformer was at first a surprise to the writer. A non zero Y_o permits an additional frequency of unity VSWR in the design band without unduly increasing the Q of the circuit. Stated in another way: The Coefficient, R_oY_o, of t^3 in B has about the same magnitude as the coefficient, $1 - R_o$, of t^2 in A (assuming

$Y_o = 0$). The particular Y_o defined in this way is denoted by Y_{oe}.

When $Y_o \neq Y_{oe}$, however, it can not be argued, as for $Y_o = 0$, that the equi-ripple solution is the optimum solution. If an attempt is made to apply the Tchebyscheff criterion in this situation, it is observed that $R_o Y_o t^3/2 \sqrt{R_o}(1 + t^2)$ is not a fixed function of t and it makes no sense to ask how well it can be approximated by a function of the form, $ct/(1 + t^2)$. On the other hand, it should be emphasized that, if Y_o is specified to be Y_{oe} in some way, it has been shown that the equi-ripple design presented above is optimum relative to any other shunted, stepped transformer of the same length. In general, however, an equi-ripple has not been proved to be optimum.

Conclusion

Clearly we have done better in minimizing P_L by putting $A \equiv 0$ and minimizing B than by putting $B \equiv 0$ and minimizing A. Undoubtedly some compromise will give the optimum performance if the value of B can further reduced without seriously increasing the value of A. For each value of Y_o and any combination of Z_1, Z_2 there exists a VSWR that is not exceeded over the design band. Denote the minimum value of this VSWR, when Z_1 and Z_2 range over all permissible values, as a function of Y_o by $\rho(Y_o)$. For the example under consideration $\rho(0) \approx 1.13$ and $\rho(.66) \approx 1.026$. It is also clear that $\rho(Y_o) > \rho(Y_{oe})$ whenever $Y_o > Y_{oe}$. Hence the minimum value of $\rho(Y_o)$, which defines the optimum shunted, stepped transformer, should be found for Y_o slightly less than Y_{oe}. The results of a careful search by computer for this minimum value of (Y_o) will be given in support of this conjecture.

NOTES